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Numerical implementation of the Crank–Nicolson/Adams–Bashforth scheme for the time-dependent Navier–Stokes equations

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SUMMARY

This article considers numerical implementation of the Crank–Nicolson/Adams–Bashforth scheme for the two-dimensional non-stationary Navier–Stokes equations. A finite element method is applied for the spatial approximation of the velocity and pressure. The time discretization is based on the Crank–Nicolson scheme for the linear term and the explicit Adams–Bashforth scheme for the nonlinear term. Comparison with other methods, through a series of numerical experiments, shows that this method is almost unconditionally stable and convergent, i.e. stable and convergent when the time step is smaller than a given constant.

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1. INTRODUCTION

Let Ω be a bounded domain in R^2 assumed to have a Lipschitz continuous boundary $\partial\Omega$ and to satisfy a further condition (A1) stated in Section 2. The following time-dependent Navier–Stokes problem is considered:

$$\begin{aligned} u_t - v\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad \operatorname{div} u = 0, \quad (x,t)\in\Omega\times(0,T] \\ u(x,0) = u_0(x), \quad x\in\Omega, \quad u(x,t)|_{\partial\Omega} = 0, \quad t\in[0,T] \end{aligned} \tag{1}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ represents the velocity vector, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force, $u_0(x)$ the initial velocity, $v>0$ the viscosity, and $T>0$ a finite time.

There are numerous works devoted to the development of efficient schemes for the Navier–Stokes equations [1–24], fully implicit, semi-implicit (semi-explicit), and explicit. Among them, high-order schemes are of more interest since first-order schemes are not sufficiently accurate for large time approximations. A key issue is the stability condition of schemes. Usually fully implicit schemes are (almost) unconditionally stable. However, at each time step, one has to solve a system of nonlinear equations. Although an explicit scheme is much easy in computation, it suffers the severely restricted time step size from stability requirement. A popular approach is based on an implicit scheme for the linear term and a semi-implicit scheme or an explicit scheme for the nonlinear term. A semi-implicit scheme for the nonlinear term results in a linear system with a variable coefficient matrix of time and an explicit treatment for the nonlinear term gives a constant matrix. Furthermore, stability and convergence conditions of the schemes have been studied by many authors, see [2, 5, 7, 10–12, 14–17].

In this article, $0<h<1$ denotes the mesh size in the spatial direction and $0<\tau = T/N<1$ denotes the time step size in the time direction. Recently, the Euler semi-implicit scheme based on the mixed finite element for solving the nonstationary Navier–Stokes equations has been studied widely. Examples include the following works:

- For a two-step scheme with a semi-implicit treatment for the nonlinear term by He and Li [5].
- For the Crank–Nicolson extrapolation scheme in which the discretization for the nonlinear term is semi-implicit by He [6].
- For the Crank–Nicolson/Adams–Bashforth scheme in which the nonlinear term is treated explicitly by Marion and Temam [11], and recently, Tone [13].
- A modified Crank–Nicolson/Adams–Bashforth scheme was proposed by Johnston and Liu [18].
- For a three-step backward extrapolating scheme (explicit for the nonlinear term) by Baker *et al*. [2].

In above works, the time step condition

$$\tau h^{-\alpha} \leqslant C_0 \tag{2}$$

for some $\alpha>0$ was imposed when a semi-implicit or an explicit scheme is applied for the nonlinear term, except the Crank–Nicolson extrapolation scheme in [6] in which a semi-implicit scheme is used for the nonlinear term. Here and after, C_0 denotes positive constant depending on the data (v, Ω, T, u_0, f).

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The second-order Crank–Nicolson/Adams–Bashforth scheme for solving the time-dependent Navier–Stokes equations consists in using a finite element pair (X_h, M_h) for the spatial discretization and the Crank–Nicolson scheme for the linear term and the Adams–Bashforth scheme for the nonlinear term for the time discretization. Under the assumption of $u_0 \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2$ with $\operatorname{div} u_0 = 0$ and $f \in L^\infty(0, T; H^1(\Omega)^2)$, f_t, $f_{tt} \in L^\infty(0, T; L^2(\Omega)^2)$, He and Sun [19] had proven that the scheme is almost unconditionally stable, i.e.

$$\|d_t u_h^m\|_{L^2}^2 + \nu \|A_h u_h^m\|_{L^2}^2 \leqslant \kappa, \quad 1 \leqslant m \leqslant N \tag{3}$$

when the condition $\tau \leqslant C_0$ is satisfied. Moreover, they also provided the optimal error estimates

$$\|u(t_m) - u_h^m\|_{L^2} \leqslant \kappa(\sigma^{-1}(t_m)\tau^2 + h^2), \quad 1 \leqslant m \leqslant N \tag{4}$$

$$\|u(t_m) - u_h^m\|_{H^1} \leqslant \kappa(\sigma^{-1/2}(t_m)\tau + h), \quad 1 \leqslant m \leqslant N \tag{5}$$

$$\|p(t_m) - p_h^m\|_{L^2} \leqslant \kappa(\sigma^{-1}(t_m)\tau + \sigma^{-1/2}(t_m)h), \quad 1 \leqslant m \leqslant N \tag{6}$$

where the finite element space pair (X_h, M_h) satisfies the approximation assumption (A3) stated in Section 3, $\sigma(t) = \min\{1, t\}$, and κ is some positive constants depending on the data (ν, Ω, T, u_0, f), and A_h is a discrete Stokes operator.

This article focuses on the numerical implementation of the Crank–Nicolson/Adams–Bashforth scheme for the two-dimensional non-stationary Navier–Stokes equations. A finite element method is applied for the spatial approximation of the velocity and pressure. The time discretization is based on the Crank–Nicolson scheme for the linear term and the explicit Adams–Bashforth scheme for the nonlinear term. Numerical examples demonstrate that the Crank–Nicolson/Adams–Bashforth scheme is almost unconditionally stable and convergent by comparing with the Euler implicit scheme and the Crank–Nicolson extrapolation scheme, respectively, when the time step τ is small than some constant C_0.

2. FUNCTIONAL SETTING OF THE NAVIER–STOKES EQUATIONS

For the mathematical setting of problem (1), the following Hilbert spaces are introduced:

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q \, \mathrm{d}x = 0 \right\}$$

The space $L^2(\Omega)^d$, $d = 1, 2, 4$, is equipped with the usual L^2-scalar product $(\cdot, \cdot)$ and L^2-norm $\|\cdot\|_{L^2}$. The spaces $H_0^1(\Omega)$ and X are equipped with their usual scalar product and equivalent norm

$$((u, v)) = (\nabla u, \nabla v), \quad \|u\|_{H_0^1} = \|\nabla u\|_{L^2}$$

Next, let the closed subset V of X be given by

$$V = \{v \in X; \operatorname{div} v = 0\}$$

and denote by H the closed subset of Y, i.e.

$$H = \{v \in Y; \operatorname{div} v = 0, v \cdot n|_{\partial\Omega} = 0\}$$

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More details on these spaces can be found in [4, 17, 21, 25] for details on these spaces. Here the Stokes operator is defined by $A = -P\Delta$, where P is the L^2-orthogonal projection of Y onto H. As mentioned above, a further assumption on Ω is presented in [7].

(A1) Assume that Ω is smooth so that the unique solution $(v, q) \in (X, M)$ of the steady Stokes problem

$$-\nu\Delta v + \nabla q = g, \quad \operatorname{div} v = 0 \text{ in } \Omega, \ \ v|_{\partial\Omega} = 0$$

for any prescribed $g \in Y$ exists and satisfies

$$\|v\|_{H^2} + \|q\|_{H^1} \leqslant c\|g\|_{L^2}$$

where $c>0$ is a generic constant depending on Ω and ν which may stand for different values at its different occurrences.

The validity of assumption (A1) is known (see [4, 17, 21, 26]) if $\partial\Omega$ is of C^2 or if Ω is a two-dimensional convex polygon. From the assumption (A1), it is well known that

$$\|v\|_{H^2} \leqslant c\|Av\|_{L^2}, \quad v \in D(A) = H^2(\Omega)^2 \cap V \tag{7}$$

$$\|v\|_{L^2} \leqslant \gamma_0 \|v\|_{H_0^1}, \ \ v \in X, \quad \|v\|_{H_0^1} \leqslant \gamma_0 \|Av\|_{L^2}, \ \ v \in D(A) \tag{8}$$

where γ_0 is a positive constant depending only on Ω [21, 25, 27].

Also, a further assumption about the prescribed data for problem is presented for problem (1).

(A2) The initial velocity $u_0(x)$ and the force $f(x,t)$ satisfy that $u_0 \in D(A)$, $f \in L^\infty(0, T; H^1(\Omega)^2)$, $f_t, f_{tt} \in L^\infty(0, T; Y)$ with

$$\|Au_0\|_{L^2} + \sup_{0\leqslant t\leqslant T} \{\|f(t)\|_{H^1} + \|f_t(t)\|_{L^2} + \|f_{tt}(t)\|_{L^2}\} \leqslant C$$

for some positive constant C. For the convenience, the following bilinear operator is introduced:

$$B(u, v) = (u \cdot \nabla)v + \tfrac{1}{2}(\operatorname{div} u)v, \quad u, v \in X$$

And the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$ are defined, respectively, by

$$a(u, v) = \nu((u, v)), \quad u, v \in X$$

and

$$d(v, q) = (q, \operatorname{div} v), \quad v \in X, \ \ q \in M$$

and a trilinear form on $X \times X \times X$ is defined by

$$\begin{aligned} b(u, v, w) &= \langle B(u, v), w\rangle_{X',X} = ((u \cdot \nabla)v, w) + \tfrac{1}{2}((\operatorname{div} u)v, w) \\ &= \tfrac{1}{2}((u \cdot \nabla)v, w) - \tfrac{1}{2}((u \cdot \nabla)w, v), \quad u, v, w \in X \end{aligned}$$

With the above notations, the variational formulation of problem (1) reads as follows: find $(u, p) \in (X, M)$ for all $t \in (0, T]$ such that for all $(v, q) \in (X, M)$,

$$(u_t, v) + a(u, v) - d(v, p) + d(u, q) + b(u, u, v) = (f, v) \tag{9}$$

$$u(0) = u_0 \tag{10}$$

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3. FINITE ELEMENT GALERKIN APPROXIMATION

Let $h>0$ be a real positive parameter. The finite element subspace (X_h, M_h) of (X, M) is characterized by $J_h = J_h(\Omega)$, a partitioning of $\bar{\Omega}$ into triangles K or quadrilaterals K, assumed to be uniformly regular as $h \to 0$.

The subspace V_h of X_h is given by

$$V_h = \{v_h \in X_h; d(v_h, q_h) = 0 \ \forall q_h \in M_h\} \tag{11}$$

Let $P_h : Y \to V_h$ denote the L^2-orthogonal projection defined by

$$(P_h v, v_h) = (v, v_h), \quad v \in Y, \quad v_h \in V_h$$

Assume that the couple (X_h, M_h) satisfies the following approximation properties:

(A3) For each $v \in H^2(\Omega)^2 \cap X$ and $q \in H^1(\Omega) \cap M$, there exist approximations $\pi_h v \in X_h$ and $\rho_h q \in M_h$ such that

$$\|v - \pi_h v\|_{H_0^1} \leqslant ch\|v\|_{H^2}, \quad \|q - \rho_h q\|_{L^2} \leqslant ch\|q\|_{H^1} \tag{12}$$

together with the inverse inequality

$$\|v_h\|_{H_0^1} \leqslant \alpha h^{-1}\|v_h\|_{L^2}, \quad v_h \in X_h \tag{13}$$

and there holds the so-called inf–sup inequality: for each $q_h \in M_h$, there exist $v_h \in X_h, v_h \neq 0$, such that

$$d(v_h, q_h) \geqslant \beta\|q_h\|_{L^2}\|v_h\|_{H_0^1} \tag{14}$$

where α and β are positive constants depending on Ω.

The standard finite element Galerkin approximation of (9)–(10) based on (X_h, M_h) reads as follows: find $(u_h, p_h) \in (X_h, M_h)$ such that for all $0 < t \leqslant T$ and $(v_h, q_h) \in (X_h, M_h)$,

$$(u_{ht}, v_h) + a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + b(u_h, u_h, v_h) = (f, v_h) \tag{15}$$

$$u_h(0) = u_{0h} = P_h u_0 \tag{16}$$

With the above statements, a discrete Stokes operator $A_h = -P_h\Delta_h$ is defined through the condition that $(-\Delta_h u_h, v_h) = ((u_h, v_h))$ for all $u_h, v_h \in X_h$. The restriction of A_h to V_h is invertible, with the inverse A_h^{-1}. Since A_h^{-1} is self-adjoint and positive definite, the 'discrete' Sobolev norms on V_h, of any order $r \in R$ can be defined by

$$\|v_h\|_r = \|A_h^{r/2} v_h\|_{L^2}, \quad v_h \in V_h$$

Under the conditions above, and with some further assumptions about the structure of the spaces X_h and M_h, it has been shown in Heywood and Rannacher [21] that

$$\|u(t) - u_h(t)\|_0 + h\|\nabla(u(t) - u_h(t))\|_0 + \sigma^{1/2}(t)h\|p(t) - p_h(t)\|_0 \leqslant \kappa h^2 \tag{17}$$

for all $t \in (0, T]$.

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4. THE CRANK–NICOLSON/ADAMS–BASHFORTH SCHEME AND RELATED NUMERICAL SCHEMES

In this section the time discretization of the finite element Galerkin approximation (15)–(16) is considered. Let $t_n = n\tau (n=0,1,\ldots,N)$, $\tau = T/N$ the time step size, and N an integer. Owing to the nature of the Adams–Bashforth scheme of three levels in time, initial value $u_h^0 = u_{0h} = P_h u_0$ is given and $(u_h^1, p_h^1) \in (X_h, M_h)$ is defined by solving the corresponding Stokes equations and the Euler-backward scheme:

$$(d_t u_h^1, v_h) + a(u_h^1, v_h) - d(v_h, p_h^1) + d(u_h^1, q_h) + b(u_h^0, u_h^0, v_h) = (f(t_1), v_h) \tag{18}$$

for all $(v_h, q_h) \in (X_h, M_h)$.

Now, the finite element solutions $(u_h^n, p_h^n) \in (X_h, M_h)$, $n=2,\ldots,N$, are defined recursively, by setting

$$\begin{aligned}(d_t u_h^n, v_h) &+ a(\bar{u}_h^n, v_h) - d(v_h, p_h^n) + d(\bar{u}_h^n, q_h) + \tfrac{3}{2} b(u_h^{n-1}, u_h^{n-1}, v_h)\\ &- \tfrac{1}{2} b(u_h^{n-2}, u_h^{n-2}, v_h) = (\bar{f}(t_n), v_h)\end{aligned} \tag{19}$$

or

$$\begin{aligned}(d_t u_h^n, v_h) &+ a(\bar{u}_h^n, v_h) - d(v_h, p_h^n) + d(\bar{u}_h^n, q_h) + b(\bar{u}_h^{n-1}, \bar{u}_h^{n-1}, v_h)\\ &+ b(d_t u_h^{n-1}, \bar{u}_h^{n-1}, v_h)\tau + b(\bar{u}_h^{n-1}, d_t u_h^{n-1}, v_h)\tau + \tfrac{1}{4} b(d_t u_h^{n-1}, d_t u_h^{n-1}, v_h)\tau^2\\ &= (\bar{f}(t_n), v_h)\end{aligned} \tag{20}$$

for all $(v_h, q_h) \in (X_h, M_h)$. Here, the following notations are introduced:

$$\bar{u}_h^n = \frac{1}{2}(u_h^n + u_h^{n-1}), \quad \bar{u}_h(t_n) = \frac{1}{2}(u_h(t_n) + u_h(t_{n-1})), \quad d_t u_h^n = \frac{1}{\tau}(u_h^n - u_h^{n-1})$$

Theorem 4.1

Suppose that the assumptions (A1)–(A3) are valid and $0<\tau<1$ satisfies the following stability condition:

$$160 c_0^2 \gamma_0^2 \nu^{-2} \kappa_2 \max\{1, \nu, \kappa_1^{1/2}\}\tau \leqslant 1 \tag{21}$$

Then the solution $(u_h^n, p_h^n) \in (X_h, M_h)$ defined by (18) and (19) satisfies

$$\|u_h^m\|_0^2 + \nu\tau \sum_{n=1}^{m} \|\bar{u}_h^n\|_1^2 \leqslant \kappa_0 \tag{22}$$

$$\|u_h^m\|_1^2 + \nu\tau \sum_{n=1}^{m} \|A_h \bar{u}_h^n\|_0^2 \leqslant \kappa_1 \tag{23}$$

$$\|d_t u_h^m\|_0^2 + \nu \|A_h u_h^m\|_0^2 + \|p_h^m\|_0^2 + \nu \|d_t u_h^m\|_1^2 \tau \leqslant \kappa_2 \tag{24}$$

for all $1 \leqslant m \leqslant N$, where $\kappa_0 \geqslant \kappa_0'$, $\kappa_1 \geqslant \kappa_1'$ and $\kappa_2 \geqslant \kappa_2'$ are some positive constants depending on the data (ν, Ω, T, u_0, f).

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Theorem 4.2
Under the assumptions of Theorem 4.1, the solution $(u_h^n, p_h^n)\in(X_h, M_h)$ defined by (18) and (19) satisfies

$$\sigma(t_m)\|u_h(t_m)-u_h^m\|_0+\sigma^{1/2}(t_m)\tau\|u_h(t_m)-u_h^m\|_1\leqslant\kappa\tau^2,\quad t_m\in(0,T] \tag{25}$$

$$\sigma(t_m)\|p_h(t_m)-p_h^m\|_0\leqslant\kappa\tau,\quad t_m\in(0,T] \tag{26}$$

The proof of Theorems 4.1 and 4.2 can be found in [19].

Then, the known Crank–Nicolson extrapolation scheme of three levels in time are recalled with the initial solution $u_h^0=u_{0h}=P_hu_0$ and $(u_h^1, p_h^1)\in(X_h, M_h)$ by solving the Stokes equations and the following Navier–Stokes equations [7]:

$$(d_tu_h^1, v_h)+a(\bar{u}_h^1, v_h)-d(v_h, p_h^1)+d(\bar{u}_h^1, q_h)+b(\bar{u}_h^1, \bar{u}_h^1, v_h)=(\bar{f}(t_1), v_h) \tag{27}$$

for all $(v_h, q_h)\in(X_h, M_h)$, respectively. Now, the finite element solutions $(u_h^n, p_h^n)\in(X_h, M_h)$, $n=2,\ldots,N$ are defined by the Crank–Nicolson extrapolation scheme [6, 7]

$$(d_tu_h^n, v_h)+a(\bar{u}_h^n, v_h)-d(v_h, p_h^n)+d(\bar{u}_h^n, q_h)+b(\tfrac{3}{2}u_h^{n-1}-\tfrac{1}{2}u_h^{n-2}, \bar{u}_h^n, v_h)=(\bar{f}(t_n), v_h) \tag{28}$$

for all $(v_h, q_h)\in(X_h, M_h)$.

Under the assumptions of Theorem 4.1, the stability and convergence results for the Crank–Nicolson/Adams–Bashforth scheme are true for the known Crank–Nicolson extrapolation scheme, see [6].

Finally, the Euler implicit scheme is recalled as follows: find $(u_h^n, p_h^n)\in(X_h, M_h)$, $n=1,\ldots,N$, such that

$$(d_tu_h^n, v_h)+a(u_h^n, v_h)-d(v_h, p_h^n)+d(u_h^n, q_h)+b(u_h^n, u_h^n, v_h)=(f(t_n), v_h) \tag{29}$$

for all $(v_h, q_h)\in(X_h, M_h)$. It has been proved that the Euler implicit scheme is unconditional stable, and convergence, with the following error estimates:

$$\|u(t_m)-u_h^m\|_0\leqslant\kappa(\tau+h^2),\quad t_m\in(0,T] \tag{30}$$

$$\|u(t_m)-u_h^m\|_1\leqslant\kappa(\tau+\sigma^{-1/2}(t_m)h),\quad t_m\in(0,T] \tag{31}$$

$$\|p(t_m)-p_h^m\|_0\leqslant\kappa(\sigma^{-1}(t_m)\tau+\sigma^{-1/2}(t_m)h),\quad t_m\in(0,T] \tag{32}$$

5. NUMERICAL RESULTS

In order to complement the theoretical results with respect to the Crank–Nicolson/Adams–Bashforth scheme for the time-dependent Navier–Stokes equations, some numerical experiments are presented in this section.

In all experiments, Ω is the unit square in R^2. The finite element discretization uses a triangle mesh with the stable P_1b-P_1 pair for the velocity and pressure. The mesh is obtained by dividing Ω into squares and then drawing a diagonal in each square. The exact solution for the velocity

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$u=(u_1, u_2)$ and pressure p are given as follows:

$$u(x,t)=(u_1(x,t), u_2(x,t)), \quad p(x,t)=10(2x_1-1)(2x_2-1)\cos(t)$$
$$u_1(x,t)=10x_1^2(x_1-1)^2x_2(x_2-1)(2x_2-1)\cos(t)$$
$$u_2(x,t)=-10x_1(x_1-1)(2x_1-1)x_2^2(x_2-1)^2\cos(t)$$

Then, the body force $f(x,t)$ is deduced from the exact solution and (1).

The first issue considers the stability of the Crank–Nicolson/Adams–Bashforth scheme for the time-dependent Navier–Stokes equations. It is well known that the Euler implicit scheme is unconditionally stable among all kinds of the schemes. To establish a reference point for the evaluation of the possible impact from the Crank–Nicolson/Adams–Bashforth scheme, the same mesh is designed for the Euler implicit scheme, the Crank–Nicolson extrapolation scheme, and the Crank–Nicolson/Adams–Bashforth scheme. Especially, the stability of the Crank–Nicolson/Adams–Bashforth scheme with the Euler implicit scheme for the large time step in stability aspect is compared.

Obviously, numbered discrete points of the velocity and pressure along the vertical line passing through the geometrical center of the cavity can be seen in a crossplot (cf. Figure 1). It is noted that $1/h=20$, $T=1s$, the reference time step $\tau=0.001$, and the solution of the given Stokes equations as an initial value. The results of the Euler implicit scheme with $\tau(=0.001)<1/h^2(=0.0025)$ are served as a standard of measurement. Then, three different time steps $\tau=0.1, 0.01$, and 0.001 are tested for the Crank–Nicolson/Adams–Bashforth scheme. Seen from Figure 1, there are no any negative effect on the large time step for the time-dependent Navier–Stokes equations by using the Crank–Nicolson/Adams–Bashforth scheme. Also, the same results are obtained by using Crank-Nicolson/Adams–Bashforth scheme with different time steps and the Euler implicit scheme with the reference time step. Accurately, further numerical study requires the results of the maximal H^1- and L^2-norm of the velocity and pressure. For means of comparison with the Euler implicit scheme, we pick out the maximal H^1- and L^2-norm of the velocity and pressure among several time iterative step on the same mesh. The results provided in Tables I–IV indicate that the Crank–Nicolson/Adams–Bashforth scheme can almost obtain the same stability as the Euler implicit scheme. In brief, results from Figure 1 and Tables I–IV show that same excellent stability of two schemes is obtained at the same time.

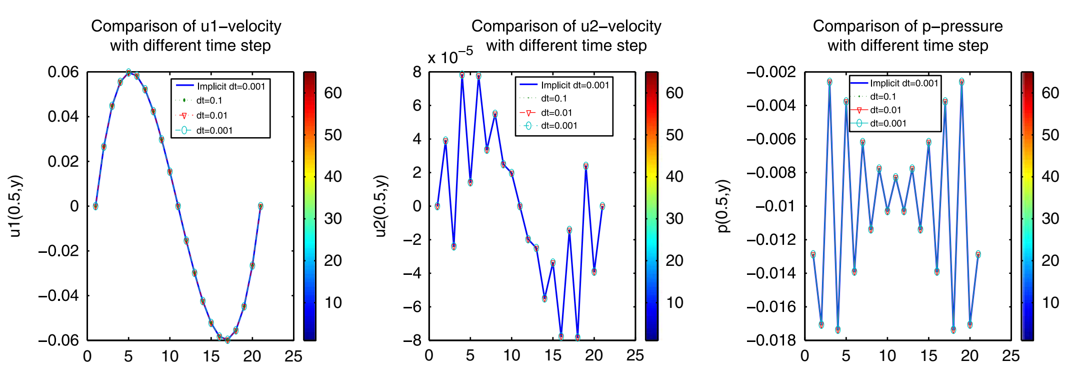

Figure 1. Comparison of u-velocity and p-pressure with different time steps τ ($\nu=0.1$ and $1/h=20$) for the Crank–Nicolson/Adams–Bashforth scheme.

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Table I. The norm $\sup_{0\leqslant t_m\leqslant 5}\|\nabla u_h^m\|_0$ of the Euler implicit scheme($v=0.1$ and $T=5$).

	τ					
$1/h$	0.5	1.0	1.1	1.2	1.3	1.4
10	0.199833	0.199832	0.199828	0.19983	0.199814	∞
20	0.139728	0.139726	0.139719	0.139722	0.134713	∞
40	0.135008	0.135006	0.134999	0.135002	0.134972	∞
60	0.13475	0.134748	0.13474	0.134743	0.134713	∞
80	0.134706	0.134704	0.134697	0.1347	0.13467	∞

Table II. The norm $\sup_{0\leqslant t_m\leqslant 5}\|p_h^m\|_0$ of the Euler implicit scheme ($v=0.1$ and $T=5$).

	τ					
$1/h$	0.5	1.0	1.1	1.2	1.3	1.4
10	3.33098	3.33098	3.33098	3.33098	3.33098	∞
20	3.33317	3.33317	3.33317	3.33317	3.33333	∞
40	3.33332	3.33332	3.33332	3.33332	3.33332	∞
60	3.33333	3.33333	3.33333	3.33333	3.33333	∞
80	3.33333	3.33333	3.33333	3.33333	3.33333	∞

Table III. The norm $\sup_{0\leqslant t_m\leqslant 5}\|\nabla u_h^m\|_0$ of the Crank–Nicolson/Adams–Bashforth scheme ($v=0.1$ and $T=5$).

	τ					
$1/h$	0.5	1.0	1.1	1.2	1.3	1.4
10	0.317589	0.483669	0.493476	0.501099	0.507129	∞
20	0.275123	0.470491	0.482361	0.491707	0.499179	∞
40	0.275123	0.471632	0.484025	0.493825	0.501689	∞
60	0.269785	0.472026	0.484516	0.4944	0.502336	∞
80	0.269617	0.47218	0.484703	0.494617	0.502579	∞

Table IV. The norm $\sup_{0\leqslant t_m\leqslant 5}\|p_h^m\|_0$ of the Crank–Nicolson/Adams–Bashforth scheme ($v=0.1$ and $T=5$).

	τ					
$1/h$	0.5	1.0	1.1	1.2	1.3	1.4
10	3.13243	2.56733	2.42305	2.2738	2.1187	∞
20	3.13645	2.56983	2.42481	2.27424	2.11937	∞
40	3.13645	2.57019	2.42512	2.27402	2.11923	∞
60	3.13723	2.57024	2.42516	2.27396	2.11920	∞
80	3.13727	2.57026	2.42518	2.27394	2.11917	∞

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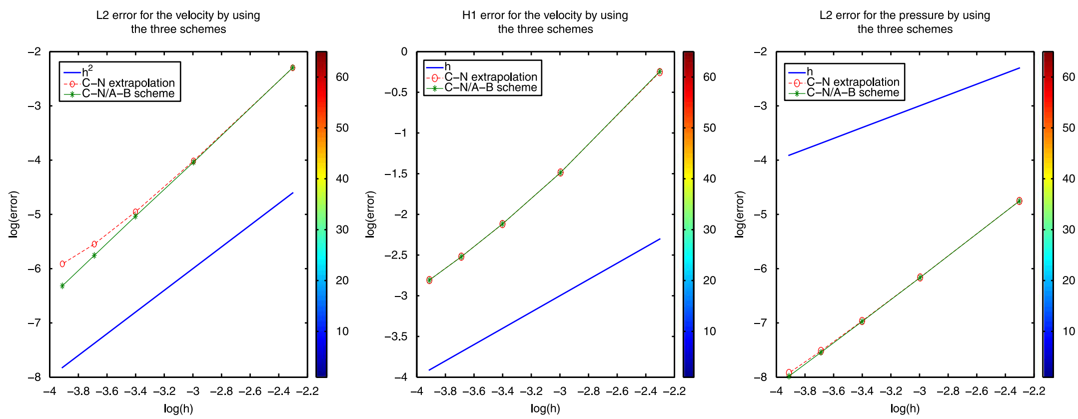

Figure 2. Rate analysis for the velocity and pressure (the Crank–Nicolson extrapolation scheme and the Crank–Nicolson/Adams–Bashforth scheme, $\nu=0.1$).

Table V. The convergence of the Crank–Nicolson extrapolation scheme ($\nu=0.1$ and $\tau=h$).

$1/h$	CPU (s)	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$
10	60.016	0.100333	0.776956	0.00861975
20	183.938	0.0180421	0.226584	0.00210659
30	242.157	0.00705652	0.120115	0.000944405
40	483.454	0.00389711	0.0804756	0.000543721
50	849.328	0.00270418	0.0603698	0.000362297

Table VI. The convergence of the Crank–Nicolson/Adams–Bashforth scheme ($\nu=0.1$ and $\tau=h$).

$1/h$	CPU (s)	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$
10	2.609	0.100709	0.784026	0.00861963
20	18.594	0.0176201	0.225956	0.00210175
30	69.313	0.00650648	0.120361	0.000935603
40	157.079	0.00316256	0.0803447	0.000529379
50	308.078	0.0018043	0.0603227	0.000341252

The second issue is to study the convergence rate of the Crank–Nicolson/Adams–Bashforth scheme for the time-dependent Navier–Stokes equations. As mentioned above, optimal order results can be obtained in [6, 7, 17]. Except for the Euler implicit scheme, the error of the temporal discretization is of second order in time step τ under the assumptions on the smoothness of the data. Obviously, the results provided in Figure 2 and Tables V–VI, indicate that it takes less *CPU* time to compute the time-dependent Navier–Stokes equations by using the Crank–Nicolson/Adams–Bashforth scheme and the Crank–Nicolson/Adams–Bashforth scheme has the same performance as the Crank–Nicolson extrapolation scheme in convergence aspect.

Driven cavity flow serves as a standard benchmark problem for the Navier–Stokes equations. Here, the driven cavity flow problem on the unit square with homogeneous boundary conditions

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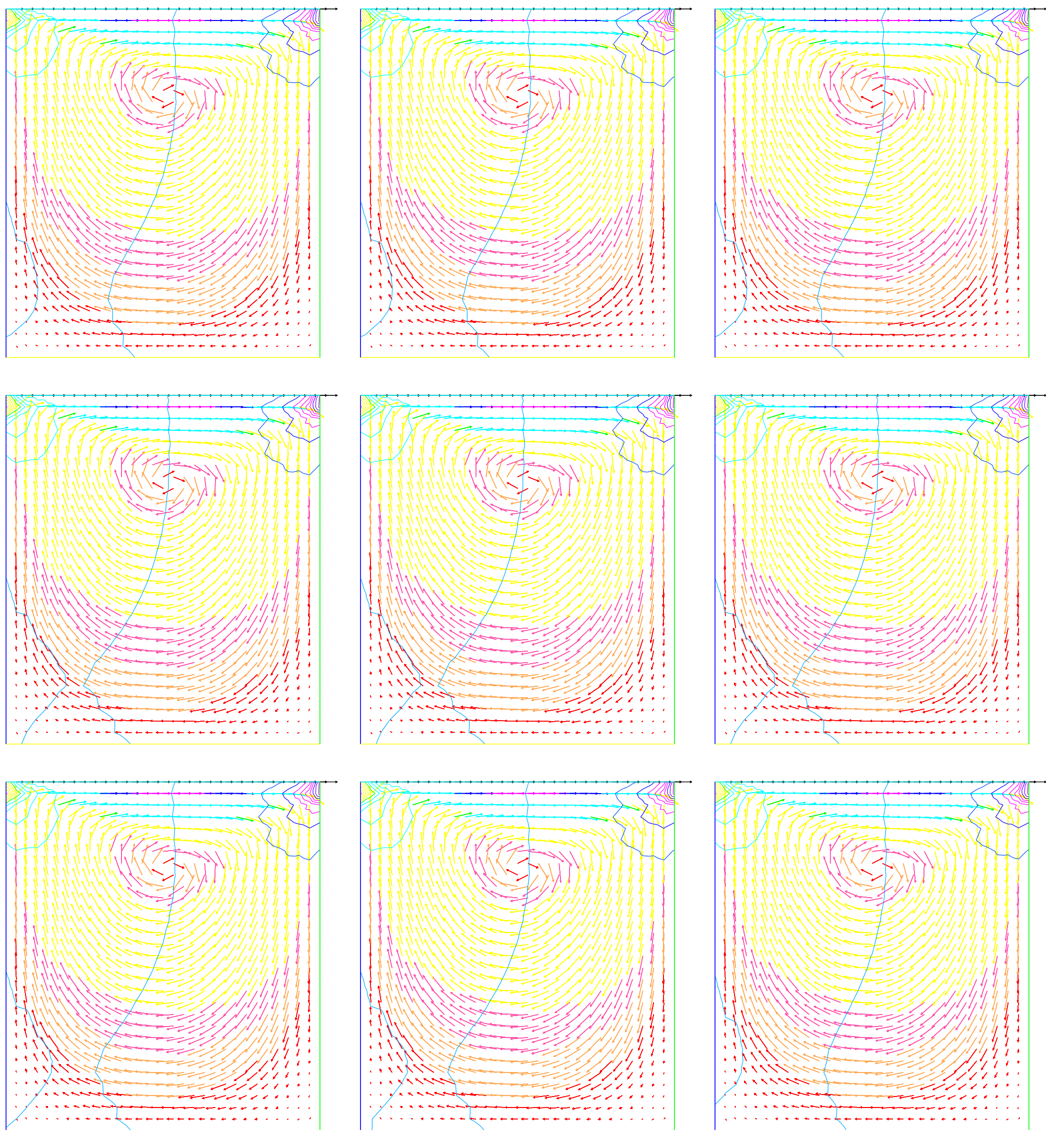

Figure 3. The velocity field and the pressure level lines for the derive cavity: the Euler implicit scheme (top), the Crank–Nicolson extrapolation scheme (middle) and the Crank–Nicolson/Adams–Bashforth scheme (bottom) ($\tau = 0.01, 0.05, 0.1$, $v = 0.1$).

on three sides, and on the top we set the tangential velocity to be $\cos(t)$ and the normal velocity to be zero. Comparison is also made between three proposed methods, the Euler implicit scheme, the Crank–Nicolson extrapolation scheme, and the Crank–Nicolson/Adams–Bashforth scheme with $v = 0.1$, $T = 1s$ and $1/h = 30$ and the various time step $\tau = 0.01, 0.05$, and 0.1 for the time-dependent Navier–Stokes equations. From Figure 3, the graphics of the three schemes are completely consistent.

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In conclusion, the Crank–Nicolson/Adams–Bashforth scheme has excellent stability and convergence with the large time step. Therefore, it has even more potential in large-scale computation.

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